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Solutions to the Lorentz force equation with fixed charge-to-mass ratio in globally hyperbolic space–times

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Abstract

We extend the classical Avez–Seifert theorem to trajectories of charged test particles with fixed charge-to-mass ratio. In particular, given two events x_0 and x_1, with x_1 in the chronological future of x_0, we find an interval $I =]-R, R[$ such that for any $q/m \in I$ there is a timelike connecting solution of the Lorentz force equation. Moreover, under the assumption that there is no null geodesic connecting x_0 and x_1, we prove that to any value of $|q/m|$ there correspond at least two connecting timelike solutions which coincide only if they are geodesics.

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1. Introduction

Let Λ be a Lorentzian manifold endowed with the metric g having signature $(+---)$, and consider a point particle of rest mass m and electric charge q, moving in the electromagnetic field F.

The equation of motion is the so called *Lorentz force equation* (cf. [5])

$$D_s\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right) = \frac{q}{mc^2}\hat{F}(x)\left[\frac{\mathrm{d}x}{\mathrm{d}s}\right], \tag{1}$$

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where $x = x(s)$ is the world line of the particle, $\mathrm{d}x/\mathrm{d}s$ is its four-velocity, $D_s(\mathrm{d}x/\mathrm{d}s)$ is the covariant derivative of $\mathrm{d}x/\mathrm{d}s$ along $x(s)$ associated to the Levi–Civita connection of g, and $\hat{F}(x)[\cdot]$ is the linear map on $T_x\Lambda$ metrically equivalent to $F(x)$, that is

$$g(x)[v, \hat{F}(x)[w]] = F(x)[v, w]$$

for any $v, w \in T_x\Lambda$.

We state the following question: Does Eq. (1) have at least one timelike future-oriented solution connecting two given events x_0 and x_1 with x_1 in the chronological future of x_0, for any charge-to-mass ratio q/m?

It is well known (see for instance [1]) that, if the manifold Λ is globally hyperbolic, the Avez–Seifert theorem gives a positive answer to the above question in the case $q = 0$ (i.e. for the *geodesic equation*).

In this paper we prove that, for an exact electromagnetic field on a globally hyperbolic manifold Λ, the answer is positive for any ratio q/m in a suitable neighborhood of $0 \in \mathbb{R}$.

Our strategy is to derive the solutions to the Lorentz force equations as projections of geodesics of a higher dimensional manifold. In this way we are able to use the techniques already developed for the geodesic equation.

This approach, has been already used for studying the Lorentz force equation in General Relativity (see for instance [4]). However, a result a la Avez and Seifert for the Lorentz force equation was still lacking.

In the preprint [2] it was proved that in a globally hyperbolic space–time Eq. (1) admits a connecting solution with a charge-to-mass ratio different from zero. That ratio, however was not fixed since the beginning.

So assume that F is an exact two-form and let ω be a potential one-form for F. Let us consider a trivial bundle $P = \Lambda \times \mathbb{R}$, $\pi : P \to \Lambda$, with the structure group $T_1 : b \in T_1$, $p = (x, y)$, $p' = pb = (x, y + b)$, and $\tilde{\omega}$ the connection one-form on P:

$$\tilde{\omega} = \mathrm{i}\left(\mathrm{d}y + \frac{e}{\hbar c}\omega\right).$$

Here y is a dimensionless coordinate on the fibre, $-e$ $(e > 0)$ is the electron charge and $\hbar = h/2\pi$, with h the Planck constant. Henceforth we will denote by $\bar{\omega}$ and $\bar{F}$, respectively the one-form $(e/\hbar c)\omega$ and the two-form $(e/\hbar c)F$. Let us endow P with the Kaluza–Klein metric

$$g^{kk} = g + a^2\tilde{\omega}^2, \tag{2}$$

or equivalently, using the notation z for the points in P and the identification $z = (x, y) \in \Lambda \times \mathbb{R}$

$$g^{kk}(z)[w, w] = g^{kk}(x, y)[(v, u), (v, u)] = g(x)[v, v] - a^2(u + \bar{\omega}(x)[v])^2$$

for every $w = (v, u) \in T_x\Lambda \times \mathbb{R}$. The positive constant a has the dimension of a length and has been introduced for dimensional consistency of definition (2). In the compactified five-dimensional Kaluza–Klein theory the fibre is isomorphic to S^1 and a represents the radius of the fifth dimension.

Let us consider the Lagrangian on P

$$L = L(z, w) : P \times TP \to \mathbb{R}, \quad L(z, w) = \tfrac{1}{2} g^{kk}(z)[w, w].$$

Fix two points p_0 and $p_1 \in P$. The geodesics on P, with respect to the Kaluza–Klein metric, connecting the points p_0 and p_1 are the critical points of the action functional

$$S = S(z) = \int_0^1 \frac{1}{2} g^{kk}(z(\lambda))[\dot{z}(\lambda), \dot{z}(\lambda)]\, \mathrm{d}\lambda,$$

defined on a suitable space of sufficiently regular curves on P, parameterized from 0 to 1, with fixed extreme points p_0 and p_1. Here $\dot{z}$ denotes the derivative of $z = z(\lambda)$ with respect to λ.

Assume that $z(\lambda) = (x(\lambda), y(\lambda))$ is a critical point for S. Since the Lagrangian L is independent of y, the following quantity p_z is conserved

$$p_z = \frac{\partial L}{\partial \dot{y}} = -a^2(\dot{y} + \bar{\omega}(x)[\dot{x}]).$$

Moreover taking variations only with respect to the variable x we obtain the following equation for $x = x(\lambda)$

$$D_\lambda \dot{x} = p_z \hat{\bar{F}}(x)[\dot{x}]. \tag{3}$$

From (3), it follows that $g(x)[\dot{x}, \dot{x}]$ is constant along x. Assume that x is non-spacelike (with respect to g) and define $C \geq 0$ such that

$$g(x)[\dot{x}, \dot{x}] = C^2.$$

Since z is a geodesic, also $g^{kk}(z)[\dot{z}, \dot{z}]$ is conserved and

$$g^{kk}(z)[\dot{z}, \dot{z}] = C^2 - \frac{p_z^2}{a^2}. \tag{4}$$

Thus the geodesic z on P is timelike iff

$$C^2 > \frac{p_z^2}{a^2}.$$

Remark 1. Of course, if z is timelike then also x is timelike, and if z is non-spacelike then also x is non-spacelike. Moreover, if z is a null geodesic, then $C^2 = p_z^2/a^2$ and x is timelike iff $p_z \neq 0$.

Remark 2. Now assume that x is timelike. The proper time for x is defined by

$$\mathrm{d}s = C\, \mathrm{d}\lambda,$$

hence if we parameterize x with respect to proper time, from (3), we get the following equation for $x = x(s)$

$$D_s\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right) = \frac{p_z}{C} \hat{\bar{F}}(x)\left[\frac{\mathrm{d}x}{\mathrm{d}s}\right] = \frac{p_z}{C} \frac{e}{\hbar c} \hat{F}(x)\left[\frac{\mathrm{d}x}{\mathrm{d}s}\right].$$

Therefore, a comparison with (1) allows us to conclude that, if we are able to find a future-oriented null geodesic for the Kaluza–Klein metric, starting from a point $p_0 = (x_0, y_0)$, arriving to a point $p_1 = (x_1, y_1)$ and having constant $p_z \neq 0$, then, recalling Remark 1, we can state that there exists a future-oriented timelike solution to (1), connecting x_0 and x_1 and having charge-to-mass ratio

$$\frac{q}{m} = \frac{p_z}{C}\frac{ec}{\hbar} = \pm\frac{aec}{\hbar},$$

with the plus sign if $p_z > 0$ and the minus sign if $p_z < 0$.

2. Statement and proof of the main theorem

In this section we state and prove our main result. In the sequel we will make large use of the notations of the book [3], which is our reference also for the necessary background on causal techniques.

Let $\mathcal{T}_{x_0,x_1}$ and $\mathcal{N}_{x_0,x_1}$ be the sets, respectively, of all the C^1, future-pointing timelike connecting curves and of all the C^1, future-pointing non-spacelike connecting curves. With *connecting curve* we mean a map x from an interval $[a, b] \subset \mathbb{R}$ to Λ such that $x(a) = x_0$ and $x(b) = x_1$ and any other map w such that $w = x \circ \lambda$ with λ a C^1 function from an interval $[c, d]$ to the interval $[a, b]$, having positive derivative.

Define

$$R = \sup_{x\in\mathcal{T}_{x_0,x_1}} \left(\frac{c^2 \int_x \mathrm{d}s}{\sup_{w\in\mathcal{N}_{x_0,x_1}} |\int_w \omega - \int_x \omega|} \right). \tag{5}$$

Notice that R does not depend on the gauge chosen, that is, it is invariant under the replacement $\omega \to \omega + \eta$, where η is an exact one-form.

We recall that a globally hyperbolic manifold is a Lorentzian manifold containing a subset (a so called *Cauchy surface*) which is intersected by every inextendible non-spacelike smooth curve precisely once.

Proposition 3. *Let (Λ, g) be a time-oriented, globally hyperbolic, Lorentzian manifold, let x_0 be a point on Λ and $x_1 \in \Lambda$ a point in the chronological future of x_0. Then*

$$R > 0.$$

Proof. Let γ be a connecting future-directed timelike geodesic whose length is

$$L = \sup_{x\in\mathcal{N}_{x_0,x_1}} \int \mathrm{d}s.$$

For the Avez–Seifert theorem, such a geodesic exists. Choose a gauge such that $\int_\gamma \omega = 0$, and define

$$M = \sup_{x\in\mathcal{N}_{x_0,x_1}} \left| \int_x \omega \right|.$$

We are going to prove that M is finite. Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_{x_0,x_1}$ be a sequence such that

$$\left|\int_{x_n} \omega\right| \to M.$$

Since (Λ, g) is globally hyperbolic, the set $C(x_0, x_1)$ of continuous non-spacelike curves connecting x_0 and x_1 is compact [3]. We recall that the topology of $C(x_0, x_1)$ is defined by saying that a neighborhood of $\eta \in C(x_0, x_1)$ consists of all the curves in $C(x_0, x_1)$ whose points in Λ lie in a neighborhood W of the points of $\eta \in \Lambda$. We can extract a subsequence, denoted again with $\{x_n\}$, such that x_n converges to a continuous non-spacelike curve x on Λ, connecting x_0 and x_1 in the topology on $C(x_0, x_1)$. Since x is compact we can cover it with m charts (U_k, ϕ_k) of the form

$$\phi_k : U_k \to \Delta^4 \subset \mathbb{R}^4 \quad \text{with} \quad \Delta =]0, b[, \tag{6}$$

where the coordinates $\{x_k^\mu\}$ are Gaussian normal coordinates

$$g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu = (\mathrm{d}x_k^0)^2 - \gamma_{ijk}(x_k^0, x_k^i)\, \mathrm{d}x_k^i\, \mathrm{d}x_k^j, \tag{7}$$

and ∂_{0k} is future-directed (the existence of a neighborhood of $p \in M$ having Gaussian coordinates follows by Lemma 4.5.2 of [3]). Here γ_{ijk} is a positive definite metric on the spacelike hypersurfaces of constant x_k^0. Moreover we can assume that $x_0 \in U_1$, $x_1 \in U_m$, and $U_i \cap U_k = \emptyset$, for any $k \neq i-1, i, i+1$. Let us introduce in U_k, the inverse of γ_{ijk}, γ_k^{-1ij}, and the function $\tilde{\omega}_k^i = \gamma_k^{-1ij}\omega_{jk}$, where ω_{jk} denote the components of ω in U_k. In U_k we consider the continuous functions ω_{0k} and $\sqrt{\tilde{\omega}_k^i \gamma_{ijk} \tilde{\omega}_k^j}$. Since x is compact, we can find a neighborhood $W \subset \bigcup_{1\le k\le m} U_k$ of x, and a constant C, such that for any k, $|\omega_{0k}| < C$ and $\sqrt{\tilde{\omega}_k^i \gamma_{ijk} \tilde{\omega}_k^j} < C$. Since x_n converges to x there is an integer number N such that, for $n > N$, $x_n \in W$.

Moreover, for any x_n, $n > N$, the strong causality condition on Λ allows us to introduce a partition $\{[\lambda_{k-1}, \lambda_k]\}_{1\le k\le m}$, $\lambda_0 = 0 < \lambda_1 < \cdots < \lambda_m = 1$, of the interval $[0, 1]$, such that $x_n([\lambda_{k-1}, \lambda_k]) \subset U_k \cap W$. Now we compute

$$\begin{aligned}\left|\int_{x_n} \omega\right| \le \int_0^1 |\omega(x_n)[\dot{x}_n]|\mathrm{d}\lambda &= \sum_{k=1}^m \int_{\lambda_{k-1}}^{\lambda_k} |\omega(x_n)[\dot{x}_n]|\mathrm{d}\lambda \\ &= \sum_{k=1}^m \int_{\lambda_{k-1}}^{\lambda_k} |\omega_{0k}\dot{x}_n^0 + \omega_{ik}\dot{x}_n^i|\mathrm{d}\lambda \\ &= \sum_{k=1}^m \int_{\lambda_{k-1}}^{\lambda_k} |\omega_{0k}\dot{x}_n^0 + \tilde{\omega}_k^i \gamma_{ijk}\dot{x}_n^j|\mathrm{d}\lambda.\end{aligned}$$

Using the Schwarz inequality we have

$$|\tilde{\omega}_k^i \gamma_{ijk} \dot{x}_n^j| \le \sqrt{(\tilde{\omega}_k^i \gamma_{ijk} \tilde{\omega}_k^j)(\dot{x}_n^s \gamma_{slk} \dot{x}_n^l)} \le C\sqrt{\dot{x}_n^i \gamma_{ijk} \dot{x}_n^j} \le C\dot{x}_n^0, \tag{8}$$

where in the last step we have used the fact that x_n is non-spacelike and future-directed. From (8) we get

$$\left|\int_{x_n} \omega\right| \leq \sum_{k=1}^{m} \int_{\lambda_{k-1}}^{\lambda_k} 2C\dot{x}_n^0 \, \mathrm{d}\lambda \leq 2Cmb < +\infty.$$

Passing to the limit on n, we conclude that M is finite. Finally, recalling the definition of R, we get

$$R \geq \frac{c^2 L}{M} > 0. \qquad \square$$

Now we are ready to state our main result.

Theorem 4. *Let (Λ, g) be a time-oriented Lorentzian manifold. Let ω be a one-form (C^2) on Λ (an electromagnetic potential) and $F = \mathrm{d}\omega$ (the electromagnetic tensor field). Assume that (Λ, g) is a globally hyperbolic manifold. Let x_1 be an event in the chronological future of x_0 and let R be defined as in (5), then there exists at least one future-oriented timelike solution to (1) connecting x_0 and x_1, for any charge-to-mass ratio satisfying*

$$\left|\frac{q}{m}\right| < R. \tag{9}$$

Before proving Theorem 4 we need some lemmas. The first is the following result, about the causal structure of the manifold P, which is contained in [2]. We report the proof for the reader convenience.

Lemma 5. *The manifold $P = \Lambda \times \mathbb{R}$ endowed with the metric (2) is a time-oriented globally hyperbolic Lorentzian manifold.*

Proof. Let V be a timelike vector field on Λ giving a time orientation. Clearly the horizontal lift of V, $(V, -\bar{\omega}[V])$, gives a time-orientation to P (henceforth we will consider P time-oriented by means of such a vector field).

Let us prove that if $z :]a, b[\to P$ $(-\infty \leq a < b \leq +\infty)$ is an inextendible smooth future-pointing non-spacelike curve, then $x(\lambda) = \pi(z(\lambda))$ is an inextendible smooth future-pointing non-spacelike curve. By contradiction let o be a future endpoint for x corresponding to $s = b$. We are going to prove that z has a future endpoint u, $\pi(u) = o$. Since z is non-spacelike we deduce that

$$a|\dot{y} + \bar{\omega}(x)[\dot{x}]| \leq \sqrt{g(x)[\dot{x}, \dot{x}]},$$

and, integrating from $c > a$ to $d < b$, we get

$$a\int_c^d |\dot{y} + \bar{\omega}(x)[\dot{x}]| \, \mathrm{d}\lambda \leq \int_c^d \sqrt{g(x)[\dot{x}, \dot{x}]} \, \mathrm{d}\lambda. \tag{10}$$

Now consider the Lorentzian distance function d on Λ associated to the metric g. Since Λ is globally hyperbolic and x is non-spacelike, the right-hand side of (10) is less than

$d(x(c), x(d)) < +\infty$. As x has future endpoint o corresponding to $\lambda = b$, $d(z(c), o) < +\infty$. So there exists the limit as $d \to b^-$ of the right-hand side of (10). Therefore the left-hand side of (10) has finite limit as $d \to b^-$. Now consider the term $\int_c^d \bar{\omega}(x)[\dot{x}]\,\mathrm{d}\lambda$. Pick a Gaussian coordinate system (U_o, φ) at o as in the proof of Proposition 3. Without loss of generality we can assume that $x(c) \in U_o$ and $x(d) \in U_o$, for any $c \le d \le b$. Denote $x(\lambda)$ by $(x^0(\lambda), x^i(\lambda))$ for any $\lambda \in [c, b]$. Since x is non-spacelike, we have

$$\gamma_{ij}(x^0, x^i)\dot{x}^i\dot{x}^j \le (\dot{x}^0)^2. \tag{11}$$

Moreover as x is future-pointing, $\dot{x}^0(\lambda) \neq 0$ on $[c, b[$, thus $x^0(\lambda)$ is strictly monotone on $[c, b[$.

Arguing as in the proof of Proposition 3, we obtain

$$\int_c^d |\bar{\omega}(x)[\dot{x}]|\,\mathrm{d}\lambda = \int_c^d |\bar{\omega}_0\dot{x}^0 + \bar{\omega}_i\dot{x}^i|\,\mathrm{d}\lambda = \int_c^d |\bar{\omega}_0\dot{x}^0 + \bar{\bar{\omega}}^i\gamma_{ij}\dot{x}^j|\,\mathrm{d}\lambda \le \int_c^d 2C\dot{x}^0\,\mathrm{d}\lambda.$$

Passing to the limit as $d \to b^-$, we conclude that $|\omega(x)[\dot{x}]|$ is integrable on $[c, b]$. As

$$\lim_{d\to b^-} \int_c^d (\dot{y} + \bar{\omega}(x)[\dot{x}])\,\mathrm{d}\lambda \in \mathbb{R},$$

we conclude that

$$\lim_{d\to b^-} y(d) - y(c) = \lim_{d\to b^-} \int_c^d \dot{y}\,\mathrm{d}\lambda \in \mathbb{R}.$$

Let $\bar{y} = \lim_{d\to b^-} y(d)$. Clearly the point $(o, \bar{y}) \in P$ is a future endpoint for z corresponding to $s = b$. This fact yields the desired contradiction.

Now, let S be a Cauchy surface for Λ, then $\tilde{S} = S \times \mathbb{R}$ is a Cauchy surface for P. Indeed $z(\lambda)$ meets $\tilde{S}$ as many times as $x(\lambda)$ meets S, and in correspondence of the same value of the parameter. Since S is a Cauchy surface for Λ, $x(\lambda)$ meets S exactly once and $z(\lambda)$ meets $\tilde{S}$ exactly once. □

Remark 6. Let $E^+(p_0) = J^+(p_0) - I^+(p_0)$, $p_0 \in P$. It is well known (see [3, pp. 112, 184]) that if $q \in E^+(p_0)$ there exists a null geodesic connecting p_0 and q.

Lemma 7. *Any globally hyperbolic Lorentzian manifold Λ is causally simple, i.e. for every compact subset K of Λ, $\dot{J}^+(K) = E^+(K)$, where $\dot{J}^+(K)$ denotes the boundary of $J^+(K)$.*

Proof. See [3, pp. 188, 207]. □

Lemma 8. *Let $p_0 = (x_0, y_0)$ and $p_1 = (x_1, y_1)$ be two points in P. Let us denote by δ the difference $y_1 - y_0$. Moreover let σ be a connecting future-oriented timelike curve. If*

$$\left|\delta + \int_\sigma \bar{\omega}\right| < \frac{\int_\sigma \mathrm{d}s}{a}, \tag{12}$$

then p_1 belongs to the chronological future of p_0.

Proof. Let λ be the affine parameter of σ such that $\mathrm{d}s/\mathrm{d}\lambda = C = \int_\sigma \mathrm{d}s$ and $\sigma(\lambda)_{|\lambda=0} = x_0$. Consider the curve on P

$$\tau = \tau(\lambda) = \left(\sigma(\lambda), y_0 + \left(\delta + \int_\sigma \bar{\omega}\right)\lambda - \int_0^\lambda \bar{\omega}[\dot{\sigma}]\,\mathrm{d}\lambda'\right).$$

Clearly $\tau(0) = p_0$, $\tau(1) = p_1$ and the following quantity is constant over τ

$$\dot{y} + \bar{\omega}[\dot{\sigma}] = \delta + \int_\sigma \bar{\omega}.$$

Moreover,

$$g^{kk}(\tau)[\dot{\tau}, \dot{\tau}] = C^2 - a^2\left(\delta + \int_\sigma \bar{\omega}\right)^2.$$

Thus, if (12) holds, τ is a timelike future-oriented curve that connects p_0 and p_1. □

Proof of Theorem 4. Let $\bar{R} = (\overline{h}/ec)R$. In Eq. (2) choose $a < \bar{R}$. There is a connecting timelike curve σ such that

$$\sup_{w\in\mathcal{N}_{x_0,x_1}} \left|\int_w \bar{\omega} - \int_\sigma \bar{\omega}\right| < \frac{\int_\sigma \mathrm{d}s}{a}. \tag{13}$$

Consider its horizontal lift σ^* having initial point $p_0 = (x_0, y_0)$. Since σ^* is timelike, its final point $\tilde{p}_1 = (x_1, \tilde{y}_1) = (x_1, y_0 - \int_\sigma \bar{\omega})$ belongs to $I^+(p_0)$. Let U be the open subset of $\mathbb{R}$ containing all the values y_1 such that $p_1 = (x_1, y_1)$ is in the chronological future of p_0. Moreover let V be the connected component of U containing $\tilde{y}_1$. Assume that V is given by $]\bar{y}_1, \hat{y}_1[$. We are going to show that $\bar{y}_1 > -\infty$ and $\hat{y}_1 < +\infty$. By contradiction, assume that for any $y_1 > \tilde{y}_1$ ($y_1 < \tilde{y}_1$), it is $p_1 = (x_1, y_1) \in I^+(p_0)$. For the Avez–Seifert theorem there is a timelike future-oriented geodesic $\alpha(\lambda) = (x(\lambda), y(\lambda))$ that connects p_0 to p_1. Here λ is the affine parameter such that $\alpha(0) = p_0$ and $\alpha(1) = p_1$. Then there exist constants C_α and p_α such that $g(x)[\dot{x}, \dot{x}] = C_\alpha^2$ and $p_\alpha = -a^2(\dot{y} + \bar{\omega})$.

Integrating the last equation from 0 to 1 gives

$$p_\alpha = -a^2\left(\delta + \int_x \bar{\omega}\right),$$

and, recalling (4)

$$g^{kk}(\alpha)[\dot{\alpha}, \dot{\alpha}] = C_\alpha^2 - a^2\left(\delta + \int_x \bar{\omega}\right)^2,$$

but we know (see the proof of Proposition 3) that $\sup_{x\in\mathcal{N}_{x_0,x_1}} |\int_x \bar{\omega}| = B < +\infty$. For $|\delta| > L/a + B$, we obtain that α is spacelike and thus a contradiction.

Now we consider the points in P, $\bar{p}_1 = (x_1, \bar{y}_1)$ and $\hat{p}_1 = (x_1, \hat{y}_1)$. By Remark 6 and Lemma 7 there exist two null geodesics $\bar{\eta} = (\bar{x}, \bar{y})$ and $\hat{\eta} = (\hat{x}, \hat{y})$ connecting p_0 to $\bar{p}_1$ and $\hat{p}_1$, respectively. By Remark 1 we know that if $p_{\bar{\eta}}, p_{\hat{\eta}} \neq 0$, then the non-spacelike curves $\bar{x}$

and $\hat{x}$ are actually timelike. Since both $\bar{p}_1$ and $\hat{p}_1$ are in $\dot{I}^+(p_0)$, from Lemma 8 and (13), we have

$$\sup_{w\in\mathcal{N}_{x_0,x_1}} \left|\int_w \bar{\omega} - \int_\sigma \bar{\omega}\right| < \left|\hat{y}_1 - y_0 + \int_\sigma \bar{\omega}\right|,$$

and analogously for $\bar{y}_1$. In particular

$$\left|\int_{\hat{x}} \bar{\omega} - \int_\sigma \bar{\omega}\right| < \left|\hat{y}_1 - y_0 + \int_\sigma \bar{\omega}\right|,$$

and analogously with the hat replaced with a bar. Recalling that $\tilde{y}_1 = y_0 - \int_\sigma \bar{\omega}$ and $\hat{y}_1 > \tilde{y}_1$ and $\bar{y}_1 < \tilde{y}_1$, we have

$$p_{\hat{\eta}} = -a^2 \left(\hat{y}_1 - y_0 + \int_{\hat{x}} \bar{\omega}\right) < 0,$$

and

$$p_{\bar{\eta}} = -a^2 \left(\bar{y}_1 - y_0 + \int_{\bar{x}} \bar{\omega}\right) > 0.$$

Therefore we have proved that there exist two timelike future-oriented connecting solutions to Eq. (1) having charge-to-mass ratios

$$\frac{q}{m} = -\frac{aec}{\hbar},$$

and

$$\frac{q}{m} = +\frac{aec}{\hbar}.$$

Since $a < \bar{R}$ is arbitrary we get the thesis. □

Theorem 9. *Let* (Λ, g) *be a time-oriented Lorentzian manifold. Let* ω *be a one-form* (C^2) *on* Λ (*an electromagnetic potential*) *and* $F = \mathrm{d}\omega$ (*the electromagnetic tensor field*). *Assume that* (Λ, g) *is a globally hyperbolic manifold. Let* x_1 *be an event in the chronological future of* x_0 *and suppose there is no null geodesic connecting* x_0 *and* x_1, *then there exist at least two future-oriented timelike solutions to* Eq. (1) *connecting* x_0 *and* x_1, *for any given value of* $|q/m|$. *The two curves coincide only if they are geodesics.*

Proof. Take an arbitrary timelike connecting curve σ and consider its horizontal lift σ^*. From σ^*, the steps of the previous proof led to two null geodesics over P. Repeating those steps here, it follows the existence of non-spacelike connecting curves $\hat{x}$ and $\bar{x}$ satisfying Eq. (3). By Remark 1, the constants of the motion $p_{\hat{\eta}}$ and $p_{\bar{\eta}}$ do not vanish, otherwise the curves $\hat{x}$ and $\bar{x}$ would be null geodesics connecting x_0 and x_1. By Remark 2, $\hat{x}$ and $\bar{x}$, parameterized by proper time are timelike future-oriented solutions of Eq. (1) having charge-to-mass ratio q/m satisfying

$$\left|\frac{q}{m}\right| = \frac{aec}{\hbar}.$$

Let them coincide, and denote them by $x = \hat{x} = \bar{x}$, then $\int_{\hat{x}} \mathrm{d}s = \int_{\bar{x}} \mathrm{d}s = C$. Moreover

$$p_{\hat{\eta}} - p_{\bar{\eta}} = -a^2(\hat{y}_1 - \bar{y}_1) \neq 0.$$

Therefore, subtracting the Lorentz force equations satisfied by both $\bar{x}$ and $\hat{x}$, we get

$$(p_{\hat{\eta}} - p_{\bar{\eta}})\hat{F}(x)\left[\frac{\mathrm{d}x}{\mathrm{d}s}\right] = 0 \Rightarrow \hat{F}(x)\left[\frac{\mathrm{d}x}{\mathrm{d}s}\right] = 0. \tag{14}$$

Substituting back this equation into the Lorentz force equation we see that x is a geodesic. Since a is arbitrary we obtain the thesis. □

Corollary 10. *Let (M, η) be the Minkowski space–time. Let ω be a one-form on M and $F = \mathrm{d}\omega$ an electromagnetic tensor field. Let x_1 be an event in the chronological future of x_0, then there exist at least two future-oriented timelike solutions to (1) connecting x_0 and x_1, for any given value of $|q/m|$. The two curves coincide only if they are geodesics.*

Proof. It follows from the fact that, in Minkowski space–time, if $x_1 \in I^+(x_0)$ there is no null geodesic connecting x_0 with x_1. □

3. Conclusions

From a physical point of view Eq. (5) shows that for sufficiently weak fields F, Theorem 4 answers affirmatively to the existence of connecting future-oriented timelike solutions of the Lorentz force equation. Indeed, under the replacement $\omega \to k\omega$, R scales as $R \to R/k$. Moreover the electron is the free particle with the maximum value of the charge-to-mass ratio, and for sufficiently small k, $e/m_e < R$.

In case the electromagnetic field is not weak, in order to have $q/m \geq R$, Eq. (5) shows that the electromagnetic "energy" $q\omega[\mathrm{d}x/\mathrm{d}s]$ should be of the same order of the rest energy mc^2. In this case quantum effects may become relevant and in particular the effect of pair creation. Theorem 9 shows that, if a pair is created at the event x_0, then at least one of the two particles has the ability, with a suitable impulse, to reach the event x_1. Notice that here we are neglecting the reciprocal electromagnetic interaction between the particles. In a strong electromagnetic field this is, however, allowed.

We conclude that in a classical regime the problem of the existence of timelike connecting solutions to the Lorentz force equations is solved. It remains open the problem of the existence of solution in a strong field F, i.e. in a quantum mechanical regime. Under these conditions we have given a partial result that can be useful when studying the consequences of the pair creations effect.

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